International Journal of Computing and Corporate Research
IJCCR India

A COMMON FIXED POINT THEOREMS IN 2-METRIC SPACES SATISFYING INTEGRAL TYPE IMPLICIT RELATION

Deo Brat Ojha
R.K.G.I.T. Ghaziabad,U.P.(INDIA)

Abstract
The aim of this paper is to prove some common fixed point theorems in 2- Metric spaces for two pairs of weakly compatible mapping satisfying integral type implicit relation. Our main result improves and extends several known results.

Keywords
2-Metric spaces, fixed point, weakly compatible mappings, compatible mappings, and implicit relation.

2000 AMS Subject Classification: 54H25,47H10.

1. Introduction
2- metric space concept was developed by Gahler[1,2,3]. On the way of development, a number of authors have studied various aspects of fixed point theory in the setting of 2-metric spaces. Iseki [4,5] is prominent in this literature which also include cho et.al.[6], Imdad et.al.[7],Murthy et.al.[8],Naidu and Prasad [9], Pathak et.al. [10]. Various authors [11,12,13] used the concepts of weakly commuting mappings compatible mappings of type(A) and (P) and weakly compatible mappings of type(A) to prove fixed point theorems in 2-metric space.

Commutativity of two mappings was weakened by Sessa [14] with weakly commuting mappings. Jungck[15] extended the class of non-commuting mappings by compatible mappings,(further Jngck & Rhodes)[16] ,Jachymski[17] ,Pant[18], Papa[19], Aliouche et.al.[20],Imdad et.al.[21], Abu-Donia & Atia[22],Popa et.al.[23] and others.

2. Preliminaries
Let X be a nonempty set. A real valued function d on X^3 is said to a 2-metric if

(D_1) to each pair of distinct points x, y in X, there exits a point $Z \in X$ *such that* $d(x,y,z) \neq 0$,

(D_2) $d(x,y,z)=0$ when at least two of x,y,z are equal,

(D_3) $d(x,y,z)=d(x,z,y)=d(y,z,x)$,

(D_4) $d(x,y,z) \leq d(x,y,u)+d(x,u,z)+d(u,y,z)$ *for all* $x,y,z,u \in X$.

The function d is called a 2-metric on set X where as the pair (X,d) stands for 2-metric space, geometrically a 2-metric $d(x,y,z)$ represents the area of a triangle with their vertices as x,y and z, As property of 2-metric d is a non-negative continuous function in any one of its three arguments but it need not be continuous in two arguments.

Definition 2.1 A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be convergent to a point $x \in X$, denoted by $\lim x_n = x,\ if\ \lim\ d(x_n, x, z) = 0\ for\ all\ z \in X$.

Definition 2.2

A sequence $\{x_n\}$ in a 2-metric space (X,d) is said to be Cauchy sequence if $\lim\ d(x_n, x_m, z) = 0\ for\ all\ z \in X$.

Definition 2.3

A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Remark 2.1[9]

Generally a convergent sequence in a 2-metric space (X,d) need not be Cauchy but every convergent sequence is Cauchy sequence whenever 2-metric d is continuous. A 2-metric d on a set X is said to be weakly continuous if every convergent sequence under d is Cauchy.

Definition 2.4[8]

Let S and T be mappings from a 2-metric space (X,d) into itself. The mappings S and T are said to be compatible if $\lim d\left(STx_n, TSx_n, z\right) = 0$ $for\ all\ z \in X$, whenever $\{x_n\}$ is a sequence is X such that $\lim Sx_n = \lim Tx_n = t$ $for\ some\ t \in X$.

Definition 2.5[22]

A pair of self mappings S and T of a 2-metric space (X,d) is said to be weakly compatible if $Sx = Tx$ ($for\ some\ x \in X$) $implies\ STx = TSx$.

Definition 2.6[8]

Let (S,T) be a pair of self mappings of a 2-metric space (X,d) .The mapping S and T are said to be compatible of type (A) if $\lim d\left(TSx_n, SSx_n, z\right) = \lim d\left(STx_n, TTx_n, z\right) = 0\ \ for\ all\ z \in X$,

When ever $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t\ \ for\ some\ t \in X$.

Definition 2.7[10]

Let (S,T) be a pair of self mappings of a 2-metric space (X,d) .Then the pair (S ,T) is said to be weakly compatible of type (A) if $\lim d\left(STx_n, TTx_n, z\right) \le \lim d\left(TSx_n, TTx_n, z\right)\ \ and$

$\lim d\left(TSx_n, SSx_n, z\right) = \lim d\left(STx_n, SSx_n, z\right) = 0\ \ for\ all\ z \in X$, where $\{x_n\}$ is a sequence in X such that $\lim Sx_n = \lim Tx_n = t\ \ for\ some\ t \in X$.

On the other hand Branciari [24] gave a fixed point result for a single mapping satisfying an analogue of Banach's contraction principle which is stated as follows,

Theorem 2.1[24]

Let (X,d) be a complete metric space, $c \in [0,1), T : X \to X$ a mapping such that, for each $x, y \in X$, $\int_0^{d(Tx,Ty)} f(t)dt \le c\int_0^{d(x,y)} f(t)dt$

where $f : R^{+} \to R^{+}$ is a Lebesgue –integrable mapping which is summable, non-negative and such that for each $s > 0.\ \int_0^s f(t)dt > 0.$ then T has a unique fixed point $z \in X$ such that, for each $x \in X, \lim_n T^n x = z.$

This result was further generalized by Abbas and Rhoades [25], Aliouche [26],Gairola and Rawat[27],Kumar et.al [28],Bryant[29].

3. Implicit Relations

Let G be the set of all continuous functions $G : R_+^6 \to R_+$ satisfying the following conditions: (G_1) G is decreasing in variables $t_2 \ldots\ldots t_6$

(G_2) There exist $h \in (0,1)$ *such that for* $u, v \geq 0$ *with*

(G_a): $\int_0^{G(u,v,v,u,u+v,0)} \varphi(t)dt \leq 0,\ implies\ u \leq h.v$,

(G_b): $\int_0^{G(u,v,u,v,0,u+v)} \varphi(t)dt \leq 0,\ implies u \leq h.v$

(G_3) $\int_0^{G(u,u,0,0,u,u)} \phi(t)\,dt > 0,\ \ for\,all\ \ u > 0$

Let ψ be the family of such functions G and $\varphi : R_+ \to R$ is a Lebesgue- integrable mapping which is summable.

Example 3.1 Let $F(t_1, t_2, \ldots\ldots t_6) = t_1 - p \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$

where $p \in (0,1)$ and $\varphi(t) = \frac{3\pi}{4(1+t)^2}.Cos\frac{3\pi t}{4(1+t)}$ *for all t in* R_+.

4. Main Results

Example 4.1 Define $G(t_1, t_2, \ldots\ldots t_6)$: $R_+^6 \to R$ *as*

$G(t_1, t_2, \ldots t_6) = t_1 - \psi\left(\max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}\right)$,Where $\psi : R^+ \to R^+$ is an increasing upper semi continuous function with $\psi(0) = 0$ *and* $\psi(t) < t$ for each t > 0 and $\varphi : R_+ \to R$ is a Lebesgue integrable mapping which is summable.

(G_1) : obvious

(G_2) : (G_a):

$\int_0^{G(u,v,v,u,u+v,0)} \frac{3\pi}{4(1+t)^2}.Cos\frac{3\pi t}{4(1+t)}dt \leq 0,$ if $u \geq v.$ *then* $Sin\frac{3\pi\{u - \phi(u)\}}{4\{1 + u - \phi(u)\}} \leq 0,$

Which implies, $u - \psi(u) = 0 \Rightarrow u = \psi(u) < u$, which is a contradiction hence, $u < v$ *and* $u \leq hv$, *where* $h \in (0,1)$.

(G_2):(G_b) : Similarly argument in (G_a).

G_3 : $\int_0^{G(u,u,0,0,u,u)} \frac{3\pi}{4(1+t)^2}.Cos\frac{3\pi t}{4(1+t)}dt > 0, : So,\ Sin\frac{3\pi\{u-\phi(u)\}}{4\{1+u-\phi(u)\}} > 0,$ for all $u > 0$

Remark 4.1 $\varphi(t) = \frac{3\pi}{4(1+t)^2}.Cos\frac{3\pi t}{4(1+t)}$ is negative for $t \in (2,\infty)$ positive for $t \in (0,2)$ and vanishes at t = 2.

Our aim in this article is to prove a common fixed theorem for a quadruple of mappings satisfying certain integral type implicit relations in 2-metric space. Which provides the tool for finding the existence of common fixed point for two pairs of weakly compatible mappings.

Now we state and prove our main result.

Proposition 4.1

Let (X,d) be a 2- metric space and $A,B,S,T : X \to X$ be four mapping satisfying the condition

$$\int_0^{G(d(Ax,By,a),d(Sx,Ty,a),d(Sx,Ax,a),d(Ty,By,a),d(Sx,By,a),d(Ty,Ax,a))} \varphi(t)dt \le 0 \qquad (4.1)$$

for all $x, y \in X$ and for all $a \in X$,where G satisfies propertiesG_1,G_2($G_{a.}$),G_2 ($G_{b.}$) and G_3 with $\varphi : R_+ \to R$ is a Lebesgue integrable mapping which is summable.

The A,B,S and T have at most one common fixed point.

Proof **:**

Let on contrary that A,B,S and T have two common fixed points u and v such that $u \ne v$. Then by (4,1), we have

$$\int_0^{G(d(Au,Bv,a),d(Su,Tv,a),d(Su,Au,a),d(Tv,Bv,a),d(Su,Bv,a),d(Tv,Au,a))} \varphi(t)dt \le 0$$

or $\int_0^{G(d(u,v,a),d(u,v,a),0,0,d(u,v,a),d(v,u,a))} \phi(t)\,dt \le 0,\ \ for\ all\ a \in X$.which contradicts ($G_3$). This provides $u = v$.

Let A,B,S and T be mappings from a 2-metric space (X,d) into itself satisfying the following condition : $A(X) \subseteq T(X)\ and\ B(X) \subseteq S(X)$ (4.2)

Since $A(X) \subseteq T(X)$, for arbitrary point $x_0 \in X$ there exits a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for the point x_1. We can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1} and\ y_{2n+1} = Ax_{2n+1} = Tx_{2n+2} : n = 0,1,2,...$ (4.3)

Lemma 4.1 If A,B,S and T be mappings from a 2-metric space (X,d) into itself which satisfy conditions (4.1) and (4.2), then

(a) $d(y_n, y_{n+1}, y_{n+2}) = 0\ \ for\ every\ n \in N$;

(b) $d\left(y_i, y_j, y_k\right) = 0 \;\; for \;\, i, j, k \in N,$ where $\{y_n\}$ is a sequence described by (4.3) and $\varphi : R_+ \to R$ is a Lebesgue integrable mapping which is summable.

Proof (a) From (4.1)

$$\int_0^{G\left\{\begin{matrix} d(Ax_{2n+2}, Bx_{2n+1}, y_{2n}), d(Sx_{2n+2}, Tx_{2n+1}, y_{2n}), d(Sx_{2n+2}, Ax_{2n+2}, y_{2n}), \\ d(Tx_{2n+1}, Bx_{2n+1}, y_{2n}), d(Sx_{2n+2}, Bx_{2n+1}, y_{2n}), d(Tx_{2n+1}, Ax_{2n+2}, y_{2n}) \end{matrix}\right\}} \varphi(t)dt \le 0$$

or, $$\int_0^{G\left\{\begin{matrix} d(y_{2n+2}, y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n+2}, y_{2n}), \\ d(y_{2n}, y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n}, y_{2n}) \end{matrix}\right\}} \varphi(t)dt \le 0$$

or, $$\int_0^{G\{d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, 0, 0\}} \varphi(t)dt \le 0$$

or, $$\int_0^{G\{d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, d(y_{2n+2}, y_{2n+1}, y_{2n}), 0, 0, d(y_{2n+2}, y_{2n+1}, y_{2n})\}} \varphi(t)dt \le 0$$

yielding there by $d\left(y_{2n+2}, y_{2n+1}, y_{2n}\right) = 0, \; due\, to \; G_b.$ Similarly using (G_a) we can show that $d(y_{2n+1}, y_{2n}, y_{2n-1}) = 0$ thus it follows that $d(y_n, y_{n+1}, y_{n+2}) = 0$ for every $n \in N.$

(b) For all $a \in X.$, let us suppose $d_n = d(y_n, y_{n+1}, y_a), n = 0,1,2,.......$ first we shall prove that $\{d_n\}$is a non-deceasing sequence in R^+, from (4.1), we have

$$\int_0^{G\left\{\begin{matrix} d(Ax_{2n}, Bx_{2n+1}, a), d(Sx_{2n}, Tx_{2n+1}, a), d(Sx_{2n}, Ax_{2n}, a), \\ d(Tx_{2n+1}, Bx_{2n+1}, a), d(Sx_{2n}, Bx_{2n+1}, a), d(Tx_{2n+1}, Ax_{2n}, a) \end{matrix}\right\}} \varphi(t)dt \le 0$$

or, $$\int_0^{G\left\{\begin{matrix} d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n-1}, y_{2n}, a), \\ d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n+1}, a), d(y_{2n}, y_{2n}, a) \end{matrix}\right\}} \varphi(t)dt \le 0$$

or, $$\int_0^{G\left\{\begin{matrix} d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), \\ d(y_{2n-1}, y_{2n+1}, y_{2n}) + d(y_{2n-1}, y_{2n}, a) + d(y_{2n+1}, y_{2n}, a), 0 \end{matrix}\right\}} \varphi(t)dt \le 0$$

or, $$\int_0^{G\left\{\begin{matrix} d(y_{2n}, y_{2n+1}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), \\ d(y_{2n-1}, y_{2n-1}, a) + d(y_{2n}, y_{2n+1}, a), 0 \end{matrix}\right\}} \varphi(t)dt \le 0$$

Implying $d_{2n} \le hd_{2n-1} < d_{2n-1}(due\ to\ (G_a))$ Similarly using (G_b), we have $d_{2n+1} \le hd_{2n}$.Thus $d_{n+1} < d_n$ for $n = 0,1,2,........$It is easy to verify $d\left(y_i, y_j, y_k\right) = 0 \;\; for \;\, i, j, k \in N.$

Lemma 4.2-Let $\{y_n\}$ be a sequence in a 2-metric space(X, d)describe by(4.3), then $d(y_n, y_{n+1}, a) = 0$ For all $a \in X$.

Proof: As in Lemma 4.1 we have $d_{2n+1} \le hd_{2n}$ and $d_{2n} \le hd_{2n-1}$. Therefore we get $d_n \le h^n d_0$. So $\lim d(y_n, y_{n+1}, a) = \lim d_n = 0.$

Lemma 4.3-Let A,B,S and T be mapping from a 2-metric space (X,d) into itself which satisfy conditions (4.1) and (4.2). Then the sequence $\{y_n\}$ describe by (4.3) is a Cauchy sequence. Where $\varphi : R^+ \to R$ is a Lebesgue-integrable mapping which is summable.

Proof: Since $\lim d(y_n, y_{n+1}, a) = 0$ by Lemma 4.2, it is sufficient to show that a sequence $\{y_{2n}\}$ or $\{y_n\}$ is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence in X , then for every $\varepsilon > 0$ there exits $a \in X$ and strictly increasing sequences $\{m_k\},\{n_k\}$ of positive integer such that $k \le n_k < m_k$ *with* $d\left(y_{2n_k-1}, y_{2m_k}, a\right) \ge \varepsilon$ *and* $d\left(y_{2n_k}, y_{2m_k-2}, a\right) < \varepsilon$. We can obtain $\lim d\left(y_{2n_k}, y_{2m_k}, a\right) = \varepsilon$, $\lim d\left(y_{2n_k}, y_{2m_k-1}, a\right) = \varepsilon$, $\lim d\left(y_{2n_k+1}, y_{2m_k}, a\right) = \varepsilon$,and $\lim d\left(y_{2n_k+1}, y_{2m_k-1}, a\right) = \varepsilon.$

Now using 4.1 we have,

or
$$\int_0^{G\left\{\begin{matrix} d\left(Ax_{2m_k}, Bx_{2n_k+1}, a\right), d\left(Sx_{2m_k}, Tx_{2n_k+1}, a\right), d\left(Sx_{2m_k}, Ax_{2m_k}, a\right), \\ d\left(Bx_{2n_k+1}, Tx_{2n_k+1}, a\right), d\left(Sx_{2m_k}, Bx_{2n_k+1}, a\right), d\left(Tx_{2n_k+1}, Ax_{2m_k}, a\right)\end{matrix}\right\}} \varphi(t)dt \le 0$$

let
$$\int_0^{G\left\{\begin{matrix} d\left(y_{2m_k}, y_{2n_k+1}, a\right), d\left(y_{2m_k-1}, y_{2n_k}, a\right), d\left(y_{2m_k-1}, y_{2m_k}, a\right), \\ d\left(y_{2n_k}, y_{2n_k+1}, a\right), d\left(y_{2m_k-1}, y_{2n_k+1}, a\right), d\left(y_{2n_k}, y_{2m_k}, a\right)\end{matrix}\right\}} \varphi(t)dt \le 0$$

$n \to \infty$, *we have*,

$$\int_0^{G(\varepsilon,\varepsilon,0,0,\varepsilon,\varepsilon)} \varphi(t) \le 0.$$

Which is a contradiction to (G_3). Therefore $\{y_{2n}\}$ is a Cauchy sequence.

Theorem 4.1-Let A,B,S and T be mappings of a 2-metric space (X,d) and $\varphi : R^+ \to R$ is a Lebesgue integrable mapping which is summable satisfy conditions (4.1) and (4.2). If one of $A(X), B(X)\ S(X)$ *or* $T(X)$ is a complete subspace of X, then

I. The pair (A,S) has a point of coincidence.

II The pair (B,T) has a point of coincidence.

Moreover, A,S,B and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.

Proof:Let $\{y_n\}$ be a sequence defined by (4.3). By Lemma (4.3),$\{y_n\}$ is a Cauchy sequence in X . Suppose that $S(X)$ is a complete subspace of X ,then the subsequence $\{y_{2n+1}\}$ which is contained in $S(X)$ must have a limit z in $S(X)$. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence $\{y_{2n+1}\}$, therefore $\{y_n\}$ also converges implying the convergence of the subsequence
$\{y_{2n}\}$, i.e.,
$\lim Ax_{2n} = \lim Bx_{2n+1} = \lim Tx_{2n+1} = \lim Sx_{2n+2} = z.$ *let* $u \in S^{-1}(z)$, *then* $Su = z.$ If $Au \ne z$, then using (4.1),we have ,

$$\int_0^{G\left\{\begin{matrix} d(Au,Bx_{2n-1},a),d(Su,Tx_{2n-1},a),d(Su,Au,a), \\ d(Tx_{2n-1},Bx_{2n-1},a),d(Su,Bx_{2n-1},a),d(Tx_{2n-1},Au,a)\end{matrix}\right\}} \varphi(t)dt \le 0$$

let $n \to \infty$, it gives, $\int_0^{G\{d(Au,z,a),d(z,z,a),d(z,Au,a),d(z,z,a),d(z,z,a),d(z,Au,a)\}} \varphi(t)dt \le 0$

hence, therefore $d(z,Au,a) = 0$ *for all* $a \in X$ (due to (G_b)),hence $z = Au = Su$.

Since $A(X) \subseteq T(X)$, *there exists* $v \in T^{-1}(z) <$ *such that* $Tv = z.$

By (4.1), we have.

$$\int_0^{G\{d(Au,Bv,a),d(Su,Tv,a),d(Su,Au,a),d(Tv,Bv,a),d(Su,Bv,a),d(Tv,Au,a)\}} \varphi(t)dt \le 0$$

or $\int_0^{G\{d(z,Bv,a),0,0,d(z,Bv,a),d(z,Bv,a),0\}} \varphi(t)dt \le 0$

Hence, therefore $d(z,Bv,a) = 0$ *for all* $a \in X$ (due to (G_a).hence $z = Bv$ .So, $Au = Su = Bv = Tv = z$, which establishes (i) and (ii).

If one assumes that T(X) is a complete subspace of X, then analogous arguments establish (i) and (ii). The remaining two cases also pertain essentially to the previous cases. If A(X) is complete, then $z \in B(X) \subseteq S(X)$. Thus in all cases (i) and (ii) are completely established.

Since A and S are weakly compatible and $Au = Su = z$,then $ASu = SAu$,which implies $Az = Sz$. By (4.1) we have

$$\int_0^{G\{d(Az,Bv,a),d(Sz,Tv,a),d(Sz,Az,a),d(Tv,Bv,a),d(Sz,Bv,a),d(Tv,Az,a)\}} \varphi(t)dt \le 0$$

or $\int_0^{G\{d(Az,z,a),(Az,z,a),0,0,d(Az,z,a),d(Az,z,a)\}} \phi(t)\,dt \le 0$

A Contradiction to (G_3) if $d(Az,z,a) > 0$. Hence $z = Az = Sz$. Since B and T are weakly compatible and Bv = Tv = z then compatible and BTv = TBv which implies $Bz = Tz$ Again By (4.1) we have,

$$\int_0^{G\{d(Az,Bz,a),d(Sz,Tz,a),d(Sz,Az,a),d(Tz,Bz,a),d(Sz,Bz,a),d(Tz,Az,a)\}} \varphi(t)dt \le 0$$

or $\int_0^{G\{d(z,Bz,a),d(z,Bz,a),0,0,d(z,Bz,a),d(z,Bz,a)\}} \varphi(t)dt \le 0$

A Contradiction to (G_3) if $d(z,Bz,a) > 0$. Hence $z = Az = Tz$. Therefore $z = Az = Sz = Bz = Tz$

Which shows that z is a common fixed point of the mappings A,B,S and T. in the view of proposition (4.1), z is the unique common fixed point of the mappings A,B,S and T.

Example 4.2[23]

Let $X = \{a,b,c,d\}$ be a finite set of R^2 equipped with natural area function on X^3. Where $a = (0,0), b = (4,0), c = (8,0), and\ d = (0,1)$. Then clearly (X,d) is a 2-metric space. Define the self mappings A,B,S and T on X as follows,

$Aa = Ab = a, Ac = Ad = b, Ba = Bb = a, Bc = Bd = a, Sa = Sb = a, Sc = c, Sd = b$ *and* $Ta = Tb = a, Tc = b, Td = c$, and $\varphi : R^+ \to R$ is a Lebesgue integrable mapping which is summable. Notice that
$A(X) = \{a,b\} \subset \{a,b,c\} = T(X)$, *and* $B(X) = \{a,b\} \subset \{a,b,c\} = S(X)$ *also*
$A(X), B(X), S(X)$ *and* $T(X)$ are complete subspace of X. the pair (A,S) is weakly compatible but not commuting as $\mathrm{ASc} = \mathrm{b} \neq \mathrm{SAc} = \mathrm{a}$. where as the pair (B,T) is commuting and hence weakly compatible.

Define $G(t_1, t_2, \ldots\ldots t_6) : R_+^6 \to R^+ as\ G(t_1, t_2, \ldots t_6) = t_1 - p\left(\max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}\right)$

Then conditions of theorem (4.1) is satisfied with $p = \frac{1}{2}$. Thus all the conditions of theorem 4.1 are satisfied and $a = (0,0)$ is a unique common fixed point of A,B,S and T and both pairs have two points of coincidence namely $a = (0,0)$ *and* $b = (4,0)$.

Theorem4.2-Let A,B,S and T be mappings from a 2-metric space (X, d) into itself. If inequality (4.1) holds for all $x, y, a \in X$ *then* $(F(S) \cap F(T)) \cap F(A) = (F(S) \cap F(T)) \cap F(B)$ and $\varphi : R^+ \to R$ is a Lebesgue integrable mapping which is summable.

Proof :Let $x \in (F(S) \cap F(T)) \cap F(A)$ .then using (4.1), we have

$$\int_0^{G\{d(Ax,Bx,a),d(Sx,Tx,a),d(Sx,Ax,a),d(Tx,Bx,a),d(Sx,Bx,a),d(Tx,Ax,a)\}} \varphi(t)dt \leq 0$$

or $\int_0^{G\{d(x,Bx,a),0,0,d(x,Bx,a),d(x,Bx,a),0\}} \phi(t)\, dt \leq 0$

Hence $d(x, Bx, a) = 0 \,\forall\, a \in X\,(due\,to\,G_a)$.*Therefore* $x = Bx$.

Thus $(F(S) \cap F(T)) \cap F(A) \subset (F(S) \cap F(T)) \cap F(B)$. Similarly using ($G_b$) we can show that,
$(F(S) \cap F(T)) \cap F(B) \subset (F(S) \cap F(T)) \cap F(A)$. Now with theorem 4.1 and 4.2, we can follows.

Theorem 4.3- Let $A, B\{T\}_{i \in N \cup \{0\}}$ be mappings of a 2-metric space (X, d) into itself such that,

I. $T_0(X) \subseteq A(X)$ *and* $T_i(X) \subseteq B(X)$,

II The pairs (T_0, B) *and* (T_i, A) $(i \in N)$ are weakly compatible.

III The inequality

$$\int_0^{G\{d(T_0x,T_iy,a),d(Ax,By,a),d(Ax,T_0x,a),d(By,T_iy,a),d(T_0x,T_iy,a),d(By,T_0x,a)\}} \varphi(t)dt \leq 0$$

where $\varphi : R^+ \to R$ is a Lebesgue integrable mapping which is summable, for each $x, y, a \leftarrow X$, $\forall\, i \in N$, *where* $G \in \psi$.(ψ as example 4.1)

Then A,B and $\{T_i\}_{i\in N\cup\{0\}}$ have a unique common fixed point in X provided one of $A(X), B(X)$ or $T_0(X)$ is a complete subspace of X .

Now as an application of theorem 4.1, we prove an integral analogue of Bryant [31] type generalized common fixed point theorem for four finite families of self mappings, which is as follows:

Theorem 4.4 Let $\{A_1, A_2,....., A_m\}$, $\{B_1, B_2,....., B_n\}$, $\{S_1, S_2....., S_p\}$ and $\{T_1, T_2....., T_q\}$ be four finite families of self mappings on a 2-metric space (X, d) with $A = A_1, A_2,....., A_m$, $B = B_1, B_2,....., B_n$, $S = S_1, S_2....., S_p$ and $T = T_1, T_2....., T_q$ so that Let A,B,S and T satisfy conditions (4.1) and (4.2) and $\varphi : R^+ \to R$ is a Lebesgue integrable mapping which is summable, If one of $A(X), B(X)\ S(X)\ or\ T(X)$ is a complete subspace of X, then

I. The pair (A,S) has a point of coincidence.
II. The pair (B,T) has a point of coincidence.

Moreover, if $A_iA_j = A_jA_i$, $S_kS_l = S_lS_k$, $B_sB_r = B_rB_s$, $T_uT_v = T_vT_u$, $A_iS_k = S_kA_i$, $and\ B_rT_v = T_vB_r$ For all $i, j \in I_1 = \{1, 2,.....m\}$, $k, l \in I_2 = \{1, 2,.....p\}$, $r, s \in I_3 = \{1, 2,.....n\}\ and\ u, v \in I_4 = \{1, 2,.....q\}$. Then for all $(i \in I_1, k \in I_2, r \in I_3\ and\ v \in I_4)$ $A_i, B_r, S_k\ and\ T_v$ have a common fixed point.

Proof :The conclusions (i) and (ii) are immediate as A,B,S and T satisfy all the conditions of Theorem 4.1. In view of pair wise commutativity of various pairs of families $\{A, S\}and\{B, T\}$,the weak compatibility of pairs (A,S) and (B,T) are immediate. Thus all the condition of theorem 4.1 (for mapping A,B,S and T) are satisfied ensuring the existence the unique common fixed point, say w. Now we need to show that w remains the fixed point of all the component maps. For this consider

$$A(A_iw) = ((A_1, A_2,....A_m)A_i)w = (A_1, A_2,....A_{m-1})((A_mA_i)w) = (A_1, A_2,....A_{m-1})(A_iA_mw)$$

$$= (A_1, A_2,....A_{m-2})(A_{m-1}A_i(A_mw)) = (A_1, A_2,....A_{m-2})(A_iA_{m-1}(A_mw)) = ...$$
$$= A_1A_i(A_2, A_3,....A_mw) = A_iA_1(A_2, A_3,....A_mw) = A_i(A_w) = A_iw.$$

Similarly, we can show that

$$A(S_kw) = S_k(Aw) = S_kw, S(S_kw) = S_k(Sw) = S_kw, S(A_iw) = A_i(Sw)$$
$$= A_iw, B(B_rw) = B_r(Bw) = B_rw$$
$$B(T_vw) = T_v(Bw) = T_vw, T(T_vw) = T_v(Tw) = T_vw,\ and\ T(B_rw) = B_r(Tw) = B_rw$$

which show that (for all i, k, r and v) A_iw and S_kw are other fixed points of the (A,S) where as B_rw and T_vw are other fixed points of the pair (B,T). Now in the view of uniqueness of the fixed point of A,B,S and T (for all i, k, r and v), we can write $w = A_iw = S_kw = B_rw = T_vw,$ which shows that w is a common fixed point of $A_i, S_k, B_r\ and\ T_v\ \ for\ all\ i, k, r\ and\ v.$

By setting $A_1 = A_2 .... = A_m = A, B_1 = B_2 .... = B_n = B, S_1 = S_2 .... = S_p = S$, $and\ T_1 = T_2 .... = T_q = T$ one can deduces the following corollary for various iterates of A,B,S and T which can also be treated as generalization of Theorem 4.1 .

Corollary 4.1

Let (A,S) and (B,T) be two commuting pairs of self mappings of 2-metric space (X, d),such that $A^m(X) \subseteq T^q(X)\ and\ B^n(X) \subseteq S^p(X)$, with $\varphi : R^+ \to R$ is a Lebesgue integrable mapping which is summable, Satisfy

$$\int_0^{G\{d(A^m x, B^n y, a), d(S^p x, T^q y, a), d(S^p x, A^m x, a), d(T^q y, B^n y, a), d(S^p x, B^n y, a), d(T^q y, A^m x, a)\}} \varphi(t)dt \leq 0 \quad (4.4)$$

for all $x, y \in X,\ and\ for\ all\ a \in X,\ where\ G \in \psi$. If one of $A^m(X), T^q(X), B^n(X)\ or\ S^p(X)$ is a complete subspace of , then A,B,S and T have a unique common fixed point.

Example 4.3:Consider $X = \{a, b, c, d\}$ is a finite subset of R^2 with $a = (0,0), b = (4,0),\ \ c = (8,0),\ \ and\ d = (0,1)$ equipped with natural area function on X^3 . Define self mappings A,B,S and T on X with $\varphi : R^+ \to R$ is a Lebesgue integrable mapping which is summable, as follows.

$Aa = Ab = Ad = a, Ac = b, Ba = Bb = Bc = a, Bd = b,$ $Ta = Tb = Tc = Td = a$,W
$and\ Sa = Sb = a, Sc = Sd = b\ and$

e see that $A^2(X) = \{a\} = T'(X)\ and\ B^2(X) = \{a\} = S^2(X)$ and the pairs (A,S) and (B,T) are commuting.

Define $G(t_1, t_2, ...t_6) : R_+^6 \to R^+ as\ G(t_1, t_2, ...t_6) = t_1 - p\left(\max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}\right)$,

where $0 < p < 1$.

Then, we verify that contraction condition 4.1 is satisfied for $A^2, B^2, S^2\ and\ T'\ as\ d(A^2 x, B^2 y, z) = d(a, a, z) = 0$ for all $x, y, z \in X$. Thus all the condition of corollary 4.1 are satisfied for A^2, B^2, S^2 and T and hence the mappings A,B,S and T have a unique common fixed point .

Even if, Theorem 4.1 is not applicable in the content of this example, as

$A(X) = \{a, b\} \not\subseteq \{a\} = T(X)\ and\ B(X) = \{a, c\} \not\subseteq \{a, b\} = S(X)$. Moreover, the contraction condition (4.1) is not satisfied for A,B,S and T. To eliminate this, we consider the case,when $\mathrm{x = c\ and\ y = a,\ we\ get\ 1 \leq p \max\{1,0,0,0,1\}} = p$

which is a contradiction to the fact that $p < 1$. Thus corollary 4.1 is slightly different to Theorem4.1.

REFERENCES

[1]. S. Gahler, 2-metrische Raume und iher topologische structur.Math.Nachr,26 (1963) , 115-148.

[2]. S. Gahler, Uber die uniforisierbarket 2-metrisches Raume, Math.Nachr.,28 (1965), 235-244.

[3]. S. Gahler, Zur geometric 2-metrischer Raume, Revue Roumaine, Math. Pures Appl.11(1966),665-667.

[4]. K. Iseki, Fixed point theorem in 2-metric spaces, Math. Sem. Notes,Kobeuni 3 (1) (1975) ,133-136.

[5]. K. Iseki, P.L. Sharma, B.K. Sharma, Contraction type mapping on 2-metric space, Math.Japonica, 21(1976),67-70.

[6]. Y.J.Cho,M.S. khan and S.L.Singh, Common fixed points of weakly commutating mappings, univ.Novom Sadu, Sb.Rd. Prirod-mat.Fak.Ser.Mat.,18(1)(1988),129-142.

[7]. M. Imdad, M.S. Kumar and M.D. Khan, A Common fixed point theorem in 2-metric spaces . Math., Japonica 36(5)(1991), 907-914.

[8]. P.P. Muthuy, S.S. Chang , Y.J.Cho and B.K.Sharma, Compatible mappings of type (A) and common fixed point theorems , Kyungpook Math, J.,32(2) (1992), 203-216.

[9]. S.V.R. Naidu and J.R. Prasad, Fixed point theorems in 2-metric spaces,Indian J. Pure Appli.Math.17(8)(1986),974-993.

[10]. H.K. Pathak, S.M. kang and J.H. Back, Weak compatible mappings of Type (A) and common fixed points, Kyungbook Math.J.,35(1995),345- 359.

[11]. Y.J. Cho., Fixed points for compatible mappings of type (A), Math. Japonica, 38(3) (1993),497-508.

[12]. D. Tan, Z. Liu and J. K. Kim, Fixed points for compatible mappings of Type (P)n 2-metric spaces, Nonlinear Funct. Anal. Appl., 8(2)(2003),215-232.

[13].W.Z. Wang, Common fixed points for compatible mappings of type (A) in 2-metric spaces, Honam Math. J., 22 (2000) , 91-97.

[14]. S.Sessa, On a weak commutativity condition in fixed point considerations, Publ.Inst. Math. (Beograd),32 (46),(1982),146-153.

[15].G. Jungck, Fixed points for non continuous non self maps on non metric spaces, Far East. J. Math. Sci., 4(2)(1996),199-215.

[16]. G. Jungck and B.E. Rhodes, Fixed points for set-valued functions without continuity, Indian J. Pure Appl., 29(3)(1998),227-238.

[17]**.** J. Jachymski, Common fixed point theorems for some families of maps, Indian J.Pure. appl. Math. 25(1995),925-937.

[18]. R. P .Pant, Common fixed point theorems of Sequences of mappings,Ganita,47(1996), 43-49.

[19].V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation,Demonstratio Math,32(1)(1999), 157-163.

[20]. A. Aliouche and A Djoudi, A general Common fixed point theorem reciprocally continuous mapping satisfying an implicit relation, AJMAA, 2(2)(2005),1-7.

[21]. M. Imdad,S. Kumar and M.S. Khan, Remarks on some fixed point theorems satisfying implicit relations. Radovi Mat., 11(2002), 135-143.

[22]. H.M. Abu-Donia and H,A. Atia, Common fixed points theorem in 2-metric spaces, Arabian J. Sci. Engg. (To appear).

[23]. V. Popa, M. Imdad and Javed Ali, Using Implicit relation to prove unified fixed point theorems in metric and 2- metric spaces, Bulletin of The Malaysian Mathematical Sciences, Accepted,1-19.

[24].A. Branciari, A Common fixed point theorem for mapping satisfying a general condition of integral type. Int. J. Math. Sci. 29(9)(2002), 531-536.

[25]. Mujahid Abbas and B.E. Rhoades, A Common fixed point theorem for hybrid pairs of occasionally weakly compatible mappings satisfying a generalized contractive condition of integral type., Fixed point theory and Applications 4 2007).
[26].A Aliouche, A Common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type. J.Math. Anal.Appl. 322(2)(2006),796-802.
[27]. U.C. Gairola and A.S. Rawat, A fixed point theorem for integral type inequality. Int. J. Math. Anal.2(15) 2008, 709-712.
[28]. S. Kumar, R. Chug and R. Kumar, Fixed point theorem compatible mappings in satisfying a contractive condition of integral type, Sochow J. Math. 33 (2) (2007) , 181-185.
[29]. V.W. Bryant, A remark on a fixed point theorem for iterated mappings.Amer. Math. Monthly ,75(1968),399-400.