A COMMON FIXED POINT THEOREMS IN 2-METRIC SPACES SATISFYING INTEGRAL TYPE IMPLICIT RELATION

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Abstract
The aim of this paper is to prove some common fixed point theorems in 2-Metric spaces for two pairs of weakly compatible mappings satisfying integral type implicit relation. Our main result improves and extends several known results.

Keywords
2-Metric spaces, fixed point, weakly compatible mappings, compatible mappings, and implicit relation.

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1. Introduction
2-Metric space concept was developed by Gahler[1,2,3]. On the way of development, a number of authors have studied various aspects of fixed point theory in the setting of 2-metric spaces. Iseki [4,5] is prominent in this literature which also include cho et.al.[6], Imdad et.al.[7], Murthy et.al.[8], Naidu and Prasad [9], Pathak et.al. [10]. Various authors [11,12,13] used the concepts of weakly commuting mappings compatible mappings of type(A) and (P) and weakly compatible mappings of type(A) to prove fixed point theorems in 2-metric space.


2. Preliminaries
Let X be a nonempty set. A real valued function d on $X^3$ is said to a 2-metric if

(D1) to each pair of distinct points $x, y$ in $X$, there exits a point $Z \in X$ such that $d(x, y, z) \neq 0$,
(D2) $d(x, y, z) = 0$ when at least two of $x, y, z$ are equal,
(D3) $d(x, y, z) = d(x, z, y) = d(y, z, x),$
(D4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

The function $d$ is called a 2-metric on set $X$ where as the pair $(X, d)$ stands for 2-metric space, geometrically a 2-metric $d(x, y, z)$ represents the area of a triangle with their vertices as $x, y$ and $z$. As property of 2-metric $d$ is a non-negative continuous function in any one of its three arguments but it need not be continuous in two arguments.
Definition 2.1 A sequence \( \{x_n\} \) in a 2-metric space \((X, d)\) is said to be convergent to a point \( x \in X \), denoted by \( \lim x_n = x \), if \( \lim d(x_n, x, z) = 0 \) for all \( z \in X \).

Definition 2.2 A sequence \( \{x_n\} \) in a 2-metric space \((X, d)\) is said to be Cauchy sequence if \( \lim d(x_n, x_m, z) = 0 \) for all \( z \in X \).

Definition 2.3 A 2-metric space \((X, d)\) is said to be complete if every Cauchy sequence in \( X \) is convergent.

Remark 2.1 Generally a convergent sequence in a 2-metric space \((X, d)\) need not be Cauchy but every convergent sequence is Cauchy sequence whenever 2-metric \( d \) is continuous. A 2-metric \( d \) on a set \( X \) is said to be weakly continuous if every convergent sequence under \( d \) is Cauchy.

Definition 2.4 Let \( S \) and \( T \) be mappings from a 2-metric space \((X, d)\) into itself. The mappings \( S \) and \( T \) are said to be compatible if \( \lim d(STx_n, TSx_n, z) = 0 \) for all \( z \in X \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim Sx_n = \lim Tx_n = t \) for some \( t \in X \).

Definition 2.5 Let \((S, T)\) be a pair of self mappings of a 2-metric space \((X, d)\).The mapping \( S \) and \( T \) are said to be compatible of type (A) if
\[
\lim d(STx_n, SSx_n, z) = \lim d(STx_n, TTx_n, z) = 0 \quad \text{for all} \quad z \in X,
\]
when ever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim Sx_n = \lim Tx_n = t \) for some \( t \in X \).

Definition 2.6 Let \((S, T)\) be a pair of self mappings of a 2-metric space \((X, d)\).Then the pair \((S, T)\) is said to be weakly compatible of type (A) if
\[
\lim d(STx_n, TTx_n, z) \leq \lim d(TSx_n, TX_n, z) \quad \text{and}
\]
\[
\lim d(TSx_n, SSx_n, z) = \lim d(STx_n, SSx_n, z) = 0 \quad \text{for all} \quad z \in X,
\]
where \( \{x_n\} \) is a sequence in \( X \) such that \( \lim Sx_n = \lim Tx_n = t \) for some \( t \in X \).

On the other hand Branciari [24] gave a fixed point result for a single mapping satisfying an analogue of Banach’s contraction principle which is stated as follows,

Theorem 2.1 Let \((X, d)\) be a complete metric space, \( c \in [0,1), T : X \to X \) a mapping such that, for each \( x, y \in X \),
\[
\int_0^{d(Tx, Ty)} f(t)dt \leq c \int_0^{d(x, y)} f(t)dt
\]
where $f : R^+ \rightarrow R^+$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $s > 0, \int_0^s f(t)dt > 0$. then $T$ has a unique fixed point $z \in X$ such that, for each $x \in X, \lim_{n \to \infty} T^n x = z$.

This result was further generalized by Abbas and Rhoades [25], Aliouche [26], Gairola and Rawat [27], Kumar et.al [28], Bryant [29].

3. Implicit Relations

Let $G$ be the set of all continuous functions $G : R^6_+ \rightarrow R_+$ satisfying the following conditions: (G$_1$) $G$ is decreasing in variables $t_2 \ldots \ldots t_6$

(G$_2$) There exist $h \in (0,1)$ such that for $u, v \geq 0$

(G$_a$) $G(u, v, u + v, v) \leq 0$, implies $u < h.v$

(G$_b$) $G(u, 0, u, 0 + v) \leq 0$, implies $u < h.v$

(G$_3$) $G(u, 0, 0, u) \leq 0$, for all $u > 0$

Let $\psi$ be the family of such functions $G$ and $\varphi : R_+ \rightarrow R$ is a Lebesgue integrable mapping which is summable.

Example 3.1 Let $F(t_1, t_2, \ldots, t_6) = t_1 - p \max \left\{ t_2, t_3, t_4, \frac{1}{2} (t_5 + t_6) \right\}$

where $p \in (0,1)$ and $\varphi(t) = \frac{3\pi}{4(1+t)} \cos \frac{3\pi t}{4(1+t)}$ for all $t$ in $R_+$.

4. Main Results

Example 4.1 Define $G(t_1, t_2, \ldots, t_6) : R^6_+ \rightarrow R$ as

$G(t_1, t_2, \ldots, t_6) = t_1 - \psi \left( \max \left\{ t_2, t_3, t_4, \frac{1}{2} (t_5 + t_6) \right\} \right)$, where $\psi : R^+ \rightarrow R^+$ is an increasing upper semi continuous function with $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$ and $\varphi : R_+ \rightarrow R$ is a Lebesgue integrable mapping which is summable.

(G$_1$) : obvious

(G$_2$) : (G$_a$): $\int_0^{G(u, v, u + v)} \frac{3\pi}{4(1+t)} \cos \frac{3\pi t}{4(1+t)} dt \leq 0$, if $u \geq v$, then $\frac{3\pi \{u - \varphi(u)\}}{4\{1+u - \varphi(u)\}} \leq 0$.

Which implies, $u - \psi(u) = 0 \Rightarrow u = \psi(u) < u$, which is a contradiction hence, $u < v$ and $u \leq hv$, where $h \in (0,1)$.

(G$_2$):(G$_b$) : Similarly argument in (G$_a$).
Remark 4.1 \( \varphi(t) = \frac{3\pi}{4(1+t)^2} \cos \frac{3\pi}{4(1+t)} \) is negative for \( t \in (2, \infty) \) positive for \( t \in (0, 2) \) and vanishes at \( t = 2 \).

Our aim in this article is to prove a common fixed theorem for a quadruple of mappings satisfying certain integral type implicit relations in 2-metric space. Which provides the tool for finding the existence of common fixed point for two pairs of weakly compatible mappings.

Now we state and prove our main result.

Proposition 4.1

Let \((X, d)\) be a 2-metric space and \(A, B, S, T: X \to X\) be four mappings satisfying the condition

\[
G_3 : \int_0^{4(a,0,0,a)} \frac{3\pi}{4(1+t)^2} \cos \frac{3\pi}{4(1+t)} \, dt > 0, \quad \text{so } \quad \sin \frac{3\pi}{4(1+u-\varphi(u))} > 0 \text{ for all } u > 0
\]

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\]

This provides \( u = v \).
(b)  \( d\left( y_i, y_j, y_k \right) = 0 \) for \( i, j, k \in N \), where \( \{y_n\} \) is a sequence described by (4.3) and \( \phi: R \rightarrow R \) is a Lebesgue integrable mapping which is summable.

Proof (a) From (4.1)
\[
\int_0^1 \left| \sum_{i,j,k=0}^N \phi(t) \right| dt \leq 0
\]
or,
\[
\int_0^1 \left| \sum_{i,j,k=0}^N \phi(t) \right| dt \leq 0
\]
\[
\int_0^1 \left| \sum_{i,j,k=0}^N \phi(t) \right| dt \leq 0
\]
yielding there by \( d\left( y_{2n+2}, y_{2n+1}, y_{2n} \right) = 0 \), due to \( G_b \). Similarly using \((G_a)\) we can show that \( d\left( y_{2n+1}, y_{2n}, y_{2n-1} \right) = 0 \) thus it follows that \( d\left( y_n, y_{n+1}, y_{n+2} \right) = 0 \) for every \( n \in N \).

(b) For all \( a \in X \), let us suppose \( d_n = d\left( y_n, y_{n+1}, y_{n+2} \right) \), \( n = 0, 1, 2, \ldots \). first we shall prove that \( \{d_n\} \) is a non-decreasing sequence in \( R^+ \), from (4.1), we have
\[
\int_0^1 \left| \sum_{i,j,k=0}^N \phi(t) \right| dt \leq 0
\]
or,
\[
\int_0^1 \left| \sum_{i,j,k=0}^N \phi(t) \right| dt \leq 0
\]
\[
\int_0^1 \left| \sum_{i,j,k=0}^N \phi(t) \right| dt \leq 0
\]
Implying \( d_{2n} \leq hd_{2n-1} < d_{2n-1} \) (due to \((G_a)\)). Similarly using \((G_b)\), we have \( d_{2n+1} \leq hd_{2n} \). Thus \( d_{n+1} < d_n \) for \( n = 0, 1, 2, \ldots \). It is easy to verify \( d\left( y_i, y_j, y_k \right) = 0 \) for \( i, j, k \in N \).

Lemma 4.2-Let \( \{y_n\} \) be a sequence in a 2-metric space \((X, d)\) describe by(4.3), then \( d\left( y_n, y_{n+1}, a \right) = 0 \) for all \( a \in X \).

Proof: As in Lemma 4.1 we have \( d_{2n+1} \leq hd_{2n} \) and \( d_{2n} \leq hd_{2n-1} \). Therefore we get \( d_n \leq h^nd_0 \). So \( \lim d\left( y_n, y_{n+1}, a \right) = \lim d_n = 0 \).
Lemma 4.3-Let A,B,S and T be mapping from a 2-metric space \((X,d)\) into itself which satisfy conditions (4.1) and (4.2). Then the sequence \(\{y_n\}\) describe by (4.3) is a Cauchy sequence. Where \(\varphi: R^+ \rightarrow R\) is a Lebesgue-integrable mapping which is summable.

Proof: Since \(\lim d(y_n, y_{n+1}, a) = 0\) by Lemma 4.2, it is sufficient to show that a sequence \(\{y_{2n}\}\) or \(\{y_n\}\) is a Cauchy sequence in \(X\). Suppose that \(\{y_{2n}\}\) is not a Cauchy sequence in \(X\), then for every \(\varepsilon > 0\) there exits \(a \in X\) and strictly increasing sequences \(\{m_k\}\), \(\{n_k\}\) of positive integer such that \(k \leq n_k < m_k\) with \(d(y_{2n_k-1}, y_{2m_k}, a) \geq \varepsilon \) and \(d(y_{2n_k}, y_{2m_k-2}, a) < \varepsilon\). We can obtain

\[
\lim\inf d(y_{2n_k}, y_{2m_k}, a) \geq \varepsilon, \quad \lim\inf d(y_{2n_k}, y_{2m_k-1}, a) = \varepsilon, \quad \lim\inf d(y_{2n_k+1}, y_{2m_k}, a) = \varepsilon, \quad \text{and}
\]

\[
\lim\inf d(y_{2n_k+1}, y_{2m_k-1}, a) = \varepsilon.
\]

Now using 4.1 we have,

\[
\int_0^\infty \left[\sum_{k=1}^\infty \left\{d(y_{2n_k}, y_{2m_k}, a) + d(y_{2n_k}, y_{2m_k-1}, a) + d(y_{2n_k+1}, y_{2m_k}, a) + d(y_{2n_k+1}, y_{2m_k-1}, a)\right]\right] \varphi(t) \, dt \leq 0
\]

or

\[
\int_0^\infty \left[\sum_{k=1}^\infty \left\{d(y_{2n_k}, y_{2m_k}, a) + d(y_{2n_k}, y_{2m_k-1}, a) + d(y_{2n_k+1}, y_{2m_k}, a) + d(y_{2n_k+1}, y_{2m_k-1}, a)\right]\right] \varphi(t) \, dt \leq 0
\]

Let \(n \rightarrow \infty\), we have,

\[
\int_0^\infty \left[\sum_{k=1}^\infty \left\{d(y_{2n_k}, y_{2m_k}, a) + d(y_{2n_k}, y_{2m_k-1}, a) + d(y_{2n_k+1}, y_{2m_k}, a) + d(y_{2n_k+1}, y_{2m_k-1}, a)\right]\right] \varphi(t) \, dt \leq 0
\]

Which is a contradiction to \((G_3)\). Therefore \(\{y_{2n}\}\) is a Cauchy sequence.

Theorem 4.1-Let A,B,S and T be mappings of a 2-metric space \((X,d)\) and \(\varphi: R^+ \rightarrow R\) is a Lebesgue integrable mapping which is summable satisfy conditions (4.1) and (4.2). If one of \(A(X), B(X), S(X)\) or \(T(X)\) is a complete subspace of \(X\), then

I. The pair \((A,S)\) has a point of coincidence.

II. The pair \((B,T)\) has a point of coincidence.

Moreover, A,S, B and T have a unique common fixed point provided both the pairs \((A,S)\) and \((B,T)\) are weakly compatible.

Proof: Let \(\{y_n\}\) be a sequence defined by (4.3). By Lemma (4.3), \(\{y_n\}\) is a Cauchy sequence in \(X\). Suppose that \(S(X)\) is a complete subspace of \(X\), then the subsequence \(\{y_{2n+1}\}\) which is contained in \(S(X)\) must have a limit \(z\) in \(S(X)\). As \(\{y_n\}\) is a Cauchy sequence containing a convergent subsequence \(\{y_{2n+1}\}\), therefore \(\{y_n\}\) also converges implying the convergence of the subsequence \(\{y_{2n}\}\), i.e.,

\[
\lim Ax_{2n} = \lim Bx_{2n+1} = \lim Tx_{2n+1} = \lim Sx_{2n+2} = z, \quad \text{let} \quad u \in S^{-1}(z), \quad \text{then} \quad Su = z. \quad \text{If} \quad Au \neq z, \quad \text{then using (4.1), we have},
\]
\[ \int_0^1 \left\{ d\left[A_x,A_y,x\right], d\left[S_x,S_y,x\right], d\left[T_x,T_y,x\right], d\left[V_x,V_y,x\right] \right\} \varphi(t) \, dt \leq 0 \]

let \( n \to \infty \), it gives,

\[ \int_0^1 \left\{ d\left[A_{x},A_{x},x\right], d\left[S_{x},S_{x},x\right], d\left[T_{x},T_{x},x\right], d\left[V_{x},V_{x},x\right] \right\} \varphi(t) \, dt \leq 0 \]

hence, therefore \( d\left(z,Au,a\right) = 0 \) for all \( a \in X \) ( due to \( G_6 \)). hence \( z = Au = Su \).

Since \( A(X) \subseteq T(X) \), there exists \( v \in T^{-1}(z) < \) such that \( Tv = z \).

By (4.1), we have,

\[ \int_0^1 \left\{ d\left[A_{x},B_{x},x\right], d\left[S_{x},T_{x},a\right], d\left(S_{x},A_{x},a\right), d\left(T_{x},B_{x},a\right), d\left(V_{x},B_{x},a\right), d\left(T_{x},A_{x},a\right) \right\} \varphi(t) \, dt \leq 0 \]

or \( \int_0^1 \left\{ d\left(z,B_{x},a\right), 0, 0, d\left(z,B_{x},a\right), d\left(z,B_{x},a\right), 0 \right\} \varphi(t) \, dt \leq 0 \)

Hence, therefore \( d\left(z,Bv,a\right) = 0 \) for all \( a \in X \) ( due to \( G_6 \)). hence \( z = Bv \). So, \( Au = Su = Bv = Tv = z \), which establishes (i) and (ii).

If one assumes that \( T(X) \) is a complete subspace of \( X \), then analogous arguments establish (i) and (ii). The remaining two cases also pertain essentially to the previous cases. If \( A(X) \) is complete, then \( z \in B(X) \subseteq S(X) \). Thus in all cases (i) and (ii) are completely established.

Since \( A \) and \( S \) are weakly compatible and \( Au = Su = z \), then \( ASu = SAu \) which implies \( Az = Sz \). By (4.1) we have

\[ \int_0^1 \left\{ d\left[A_{z},T_{z},a\right], d\left(S_{z},A_{z},a\right), d\left(T_{z},B_{z},a\right), d\left(V_{z},B_{z},a\right), d\left(T_{z},A_{z},a\right) \right\} \varphi(t) \, dt \leq 0 \]

or \( \int_0^1 \left\{ d\left(A_{z},z,a\right), 0, 0, d\left(A_{z},z,a\right), d\left(A_{z},z,a\right) \right\} \varphi(t) \, dt \leq 0 \)

A Contradiction to \( G_3 \) if \( d\left(A_{z},z,a\right) > 0 \). Hence \( z = Az = Sz \). Since \( B \) and \( T \) are weakly compatible and \( Bv = Tv = z \) then compatible and \( BTv = TBv \) which implies \( Bz = Tz \). Again By (4.1) we have,

\[ \int_0^1 \left\{ d\left(A_{z},B_{z},a\right), d\left(S_{z},T_{z},a\right), d\left(S_{z},A_{z},a\right), d\left(T_{z},B_{z},a\right), d\left(S_{z},B_{z},a\right), d\left(T_{z},A_{z},a\right) \right\} \varphi(t) \, dt \leq 0 \]

or \( \int_0^1 \left\{ d\left(z,B_{z},a\right), d\left(z,B_{z},a\right), 0, 0, d\left(z,B_{z},a\right), d\left(z,B_{z},a\right) \right\} \varphi(t) \, dt \leq 0 \)

A Contradiction to \( G_1 \) if \( d\left(z,Bz,a\right) > 0 \). Hence \( z = Az = Tz \). Therefore \( z = Az = Sz = Bz = Tz \)

Which shows that \( z \) is a common fixed point of the mappings \( A,B,S \) and \( T \). in the view of proposition (4.1), \( z \) is the unique common fixed point of the mappings \( A,B,S \) and \( T \).

Example 4.2[23]

Let \( X = \{a,b,c,d\} \) be a finite set of \( R^2 \) equipped with natural area function on \( X^3 \).

Where \( a = (0,0), b = (4,0), c = (8,0), \text{ and } d = (0,1) \). Then clearly \( (X,d) \) is a 2-

dimensional space. Define the self mappings \( A,B,S \) and \( T \) on \( X \) as follows,
\[ Aa = Ab = a, Ac = Ad = b, Ba = Bb = a, Bc = Bd = a, Sa = Sb = a, Sc = c, Sd = b \text{ and } Ta = Tb = a, Tc = b, Td = c \], and \( \phi : R^+ \to R \) is a Lebesgue integrable mapping which is summable. Notice that 
\[ A(X) = \{a, b\} \subseteq \{a, b, c\} = T(X), \text{ and } B(X) = \{a, b\} \subseteq \{a, b, c\} = S(X) \text{ also } \]
\[ A(X), B(X), S(X) \text{ and } T(X) \text{ are complete subspace of } X. \text{ the pair } (A, S) \text{ is weakly compatible but not commuting as } ASc = b \neq SAc = a. \text{ where as the pair } (B, T) \text{ is commuting and hence weakly compatible.} \]

Define \( G(t_1, t_2, \ldots, t_6) : R_+^6 \to R^+ \) as \( G(t_1, t_2, \ldots, t_6) = t_1 - p \left( \max \left\{ t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6) \right\} \right) \)

Then conditions of theorem (4.1) is satisfied with \( p = \frac{1}{2} \). Thus all the conditions of theorem 4.1 are satisfied and \( a = (0, 0) \) is a unique common fixed point of A, B, S and T and both pairs have two points of coincidence namely \( a = (0, 0) \) and \( b = (4, 0) \).

Theorem 4.2 - Let A, B, S and T be mappings from a 2-metric space \((X, d)\) into itself. If inequality (4.1) holds for all \( x, y \in X \) then 
\[ (F(S) \cap F(T)) \cap F(A) = (F(S) \cap F(T)) \cap F(B) \] and \( \phi : R^+ \to R \) is a Lebesgue integrable mapping which is summable.

Proof: Let \( x \in (F(S) \cap F(T)) \cap F(A) \) then using (4.1), we have
\[
\int_{0}^{\infty} G[d(Ax, Bx, a), d(Bx, Tx, a), d(Bx, Ax, a), d(Bx, Bx, a), d(Bx, Bx, a), d(Tx, Ax, a)] \phi(t) dt \leq 0
\]

or
\[
\int_{0}^{\infty} G[d(x, Bx, a), 0, 0, d(x, Bx, a), d(x, Bx, a), 0] \phi(t) dt \leq 0
\]

Hence \( d(x, Bx, a) = 0 \forall x \in X \) (due to \( G_{a} \)). Therefore \( x = Bx \). Thus \( (F(S) \cap F(T)) \cap F(A) \subseteq (F(S) \cap F(T)) \cap F(B) \). Similarly using \( G_{a} \) we can show that, \( (F(S) \cap F(T)) \cap F(B) \subseteq (F(S) \cap F(T)) \cap F(A) \). Now with theorem 4.1 and 4.2, we can follows.

Theorem 4.3 - Let \( A, B \{T\}_{i \in N} \) be mappings of a 2-metric space \((X, d)\) into itself such that,
\[ 1. T_0(X) \subseteq A(X) \text{ and } T_0(X) \subseteq B(X), \]
\[ 2. \text{The pairs } (T_0, B) \text{ and } (T_i, A) \text{ } (i \in N) \text{ are weakly compatible.} \]
\[ 3. \text{The inequality } \int_{0}^{\infty} G[d(T_0, x, y, a), d(Ax, By, a), d(AX, Ax, a), d(BX, By, a), d(BX, BX, a), d(TX, AT, a)] \phi(t) dt \leq 0 \]

where \( \phi : R^+ \to R \) is a Lebesgue integrable mapping which is summable, for each \( x, y, a \leftarrow X, \forall i \in N, \text{ where } G \in \psi. (\psi \text{ as example 4.1}) \)
Then A, B and \( \{ T_i \}_{i \in N \cup \{0\}} \) have a unique common fixed point in \( X \) provided one of
\( A(X), B(X) \) or \( T_i(X) \) is a complete subspace of \( X \).

Now as an application of theorem 4.1, we prove an integral analogue of Bryant [31] type generalized common fixed point theorem for four finite families of self mappings, which is as follows:

**Theorem 4.4** Let \( \{ A_1, A_2, \ldots, A_m \}, \{ B_1, B_2, \ldots, B_n \}, \{ S_1, S_2, \ldots, S_p \}, \) and
\( \{ T_1, T_2, \ldots, T_q \} \) be four finite families of self mappings on a 2-metric space \((X, d)\) with \( A = A_1, A_2, \ldots, A_m, B = B_1, B_2, \ldots, B_n, S = S_1, S_2, \ldots, S_p, \) and
\( T = T_1, T_2, \ldots, T_q \) so that Let A,B,S and T satisfy conditions (4.1) and (4.2) and \( \varphi: \mathbb{R}^+ \to \mathbb{R} \) is a Lebesgue integrable mapping which is summable. If one of
\( A(X), B(X), S(X) \) or \( T(X) \) is a complete subspace of \( X \), then
I. The pair \( (A, S) \) has a point of coincidence.
II. The pair \( (B, T) \) has a point of coincidence.

Moreover, if \( A_j A_j = A_k A_k, S_j S_j = S_k S_k, B_j B_j = B_k B_k, T_i T_i = T_k T_k, A_j S_j = S_k A_k, \) and
\( B_j T_j = T_k B_k \) for all \( i, j \in I_1 = \{1, 2, \ldots, m\}, \) \( k, l \in I_2 = \{1, 2, \ldots, n\}, \)
\( r, s \in I_3 = \{1, 2, \ldots, p\} \) and \( u, v \in I_4 = \{1, 2, \ldots, q\} \). Then for all
\( (i \in I_1, k \in I_2, r \in I_3 \) and \( v \in I_4) \) \( A_i, B_j, S_k \) and \( T_v \) have a common fixed point.

**Proof:** The conclusions (i) and (ii) are immediate as A,B,S and T satisfy all the conditions of Theorem 4.1. In view of pair wise commutativity of various pairs of families \( \{A,S\} \) and \( \{B,T\} \), the weak compatibility of pairs \( (A,S) \) and \( (B,T) \) are immediate. Thus all the condition of theorem 4.1 (for mapping A,B,S and T) are satisfied ensuring the existence the unique common fixed point, say w. Now we need to show that w remains the fixed point of all the component maps. For this consider
\[ A \left( A_i w \right) = \left( \left( A_1, A_2, \ldots, A_{m-2} \right) \left( A_{m-1} A_m \right) \right) w = \left( A_1, A_2, \ldots, A_{m-1} \right) \left( \left( A_m A_1 \right) w \right) = \left( A_1, A_2, \ldots, A_{m-1} \right) \left( A_i A_{m} \right) w \]
\[ = \left( A_1, A_2, \ldots, A_{m-1} \right) \left( A_{m-1} A_{m} \right) w = \left( A_1, A_2, \ldots, A_{m-2} \right) \left( A_{m-1} \ A_{m} \right) w = \ldots \]
which shows that \( w \) is a common fixed point of \( A_i, S_k, B_j, \) and \( T_v \) for all \( i, k, r, \) and \( v \).
By setting \( A_1 = A_2 = \ldots = A_p = A, B_1 = B_2 = \ldots = B_p = B, S_1 = S_2 = \ldots = S_p = S \), and \( T_1 = T_2 = \ldots = T_q = T \) one can deduces the following corollary for various iterates of \( A,B,S \) and \( T \) which can also be treated as generalization of Theorem 4.1.

Corollary 4.1

Let \( (A,S) \) and \( (B,T) \) be two commuting pairs of self mappings of 2-metric space \( (X,d) \), such that \( A^n(X) \subseteq T^q(X) \) and \( B^n(X) \subseteq S^p(X) \), with \( \phi : R^+ \rightarrow R \) is a Lebesgue integrable mapping which is summable, satisfy

\[
\int_0^1 \phi(t) dt \leq 0 \quad (4.4)
\]

for all \( x, y \in X \), and for all \( a \in X \), where \( G \in \psi \). If one of \( A^n(X), T^q(X), B^n(X) \) or \( S^p(X) \) is a complete subspace of \( X \), then \( A,B,S \) and \( T \) have a unique common fixed point.

Example 4.3: Consider \( X = \{a,b,c,d\} \) is a finite subset of \( R^2 \) with \( a = (0,0), b = (4,0), c = (8,0), \) and \( d = (0,1) \) equipped with natural area function on \( X^2 \). Define self mappings \( A,B,S \) and \( T \) on \( X \) with \( \phi : R^+ \rightarrow R \) is a Lebesgue integrable mapping which is summable, as follows.

\[
Aa = Ab = Ad = a, Ac = b, Ba = Bb = Bc = a, Bd = b, \quad Ta = Tb = Tc = Td = a, W
\]

\[
e \quad Sa = Sb = a, Sc = Sd = b \quad \text{and} \quad Sb = a, Sc = Sd = b \quad \text{and}
\]

we see that \( A_2(X) = \{a\} = T^1(X) \) and \( B^2(X) = \{a\} = S^2(X) \) and the pairs \( (A,S) \) and \( (B,T) \) are commuting.

Define \( G(t_1,t_2,\ldots,t_6) : R_+^6 \rightarrow R^+ \) as \( G(t_1,t_2,\ldots,t_6) = t_4 - \frac{1}{2}(t_5 + t_6) \), where \( 0 < p < 1 \).

Then, we verify that contraction condition 4.1 is satisfied for \( A^2, B^2, S^2 \) and \( T \) as \( d(A^2x,B^2y,z) = d(a,a,z) = 0 \) for all \( x, y, z \in X \). Thus all the condition of corollary 4.1 are satisfied for \( A^2, B^2, S^2 \) and \( T \) and hence the mappings \( A,B,S \) and \( T \) have a unique common fixed point.

Even if, Theorem 4.1 is not applicable in the context of this example, as \( A(X) = \{a,b\} \not\subseteq \{a\} = T(X) \) and \( B(X) = \{a,c\} \not\subseteq \{a,b\} = S(X) \). Moreover, the contraction condition (4.1) is not satisfied for \( A,B,S \) and \( T \). To eliminate this, we consider the case, when \( x = c \) and \( y = a \), we get \( 1 \leq max\{1,0,0,0,1\} = p \) which is a contradiction to the fact that \( p < 1 \). Thus corollary 4.1 is slightly different to Theorem 4.1.

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