ISSN (Online) 2249 - 054 X

International Journal of Computing and Corporate Research

Specialized and Refereed Journal for Research Scholars, Academicians, Engineers and Scientists

Volume 1 Issue 2 September 2011

PSEUDOINVEX FUNTIONS AND PSEUDOINMON OPERATORS

Deo Brat Ojha[1]

Abstract

The notions of a pseudoinvex functions and pseudoinmon operator are introduced, with cyclic pseudoinmon operator also. In this paper, we are studying the relation among pseudoinvex functions. Pseudoinmon and cyclic pseudoinmon operators.

Key words: Pseudoinvex functions, Pseudoinmon operator, Cyclic Pseudoinmon operator, Pseudomonotone, Cyclic Pseudomonotone.

1991 Mathematics subject classification: 26A51, 26E15, 47H05, 90A40, 90C26.

1.Introduction:

The notion of a (not necessary single-valued) monotone operator was first proposed by Kachurovaskii[3], interested reader may see[4,5] , cyclically monotone operator was introduced and investigated byRockafellar[6-9].

For a given smooth function f on a convex domain characterization of its convexity in terms of monotonicity or cyclically monotonicity of grad f are known as follows:

f is convex $\Leftrightarrow$ grad f is monotone $\Leftrightarrow$ grad f is cyclically monotone ----- (A)

Levin [12], get similar characterization theorems connecting

[1] R.K.G.I.T. Gzb.(INDIA)

ISSN (Online) 2249 - 054 X

International Journal of Computing and Corporate Research

Specialized and Refereed Journal for Research Scholars, Academicians, Engineers and Scientists

Volume 1 Issue 2 September 2011

f is quasiconvex $\Leftrightarrow$ grad f is quasimonotone $\Leftrightarrow$ grad f is quasicyclically monotone- (B)

The term inmonicity was introduced in [13] and showed that

Pseudoinmonicity $\Rightarrow$ quasiinmonicity , but converse is not true. ---------- (C)

On the other hand invexity of a real-valued function , the relation in

Pseudoinvexity $\Rightarrow$ quasiinvexity , but converse is not true. ---------- (D)

In [12] , Levin showed that such operators are closely related to the so-called demand functions in mathematical economics. It is worth noting that monotonicity has played a very important role in the existence and solution methods of variational inequality problems. A plenty of applications of solutions of variational inequality problems are present in the convexity literature .

Our aim of present paper is to get characterization theorems connecting pseudoinvex functions with pseudoinmon and cyclically pseudoinmon operators.

2.Notations and Preliminaries:

Let W be a Hausdroff locally convex space, X a convex set in it, and w^* the dual space.

Definition 2.1 [12] An operator $f: X \to w^*$ is Pseudomonotone if

$$< y-x, f(x) > \geq 0 \Rightarrow < (y-x), f(y) > \geq 0 \qquad \ldots\ldots \ldots(1.1)$$

$$\forall x, y \in X$$

Definition 2.2 [12] An operator $f: x \to w^*$ is cyclically pseudomonotone if

$$< x_{i+1} - x_i : f(x_i) > \quad > 0 \qquad \forall\, i = 1,2,\ldots,k$$

ISSN (Online) 2249 - 054 X
International Journal of Computing and Corporate Research
Specialized and Refereed Journal for Research Scholars, Academicians, Engineers and Scientists

Volume 1 Issue 2 September 2011

$$\Rightarrow < x_{j+1} - x_j : f(x_j) > \quad < 0 \qquad \text{for some} \quad j = 1,2,...,k$$

Definition 2.3 [13] A differentiable operator *f* on a subset $X \subseteq R^n$ is a Pseudoinvex with respect to $\eta : XxX \to X$ if for every pair of distinct points $x,y \in X$

$$\eta(y,x)^T f^{'}(x) \geq 0 \Rightarrow f(y) \geq f(x).$$

Definition 2.4 [13] Let X$\subseteq R^n$ be an invex set with respect to $\eta : XxX \to X$ and $f : X \to R$ is pseudoinmon on X if $\forall x, y \in X$.

$$\eta(y,x)^T f(x) \geq 0 \Rightarrow \eta(y,x)^T f(y) \leq 0.$$

The goal of the present paper is to get characterization theorem connecting Pseudo invex functions, Pseudo inmon and cyclically Pseudoinmon operators on an invex domain.

3. Characterization Theorem:

Let X be an invex set in a real vector space.

Definition 3.1 Let X$\subseteq R^n$ be an invex set with respect to $\eta : XxX \to X$, $f : X \to R$ is said to cyclically pseudoinmon with respect to $\exists \ i = \{1,2,....,k\}$, $\exists \ f(x_i) :< f(x_i), n(x_{i+1}, x_i) > \geq 0$

$$< f(x_{i+1}), n(x_i . x_{i+1}) > \leq 0$$

Let us define.

$$\phi(f,x,y;t) = f(x + tn(y,x)) \qquad \ldots (3.1)$$

ISSN (Online) 2249 - 054 X

International Journal of Computing and Corporate Research

Specialized and Refereed Journal for Research Scholars, Academicians, Engineers and Scientists

Volume 1 Issue 2 September 2011

Theorem 3.1 Let X be an invex open set in a Hausdroff locally convex space w, and suppose that a function $f : X \to R$ has two properties as follows:

(i) f is $G\hat{a}teaux$ differentiable on X, i.e. every $x \in X$, there exist an element grad$f(x) \in w^*$ such that

$$\lim_{\Delta t \to 0} \frac{f(x + t\eta(y,x)) - f(x)}{t} = <\eta(y,x), grad\ f(x)> \ for \text{ all } \eta(y,x) \in E.$$

(ii) for every $x,y \in X$, the function $\phi(f,x,y;.)$ given by (3.1) is absolutely continuous on [0,1].

The following assertions are then equivalent:

(a) f is pseudoinvex,

(b) the operator grad f is cyclically pseudoinmon

(c) The operator grad f is pseudoinmon.

Before to pass on to a more general characterization theorem, let us formulate two assumption (D_1) and (D_2), on a function $f : X \to R$, where X is an invex set in a real vector space.The assumptions are expressed in terms of the functions $\phi(f,x,y;.)$ (see (3.1)) as follows:

(D_1) for every $x,y \in X$ and every $t, 0 \le t \le 1$, there exist the right derivative

$$D\ \phi(f,x,y;t) = \lim_{\Delta t \to 0} \frac{\phi(f,x,y;t+\Delta t) - \phi(f,x,y;t)}{\Delta t} \quad \text{--------------- (3.2)}$$

(D_2)for every $x,y \in X$, $\phi(f,x,y;.)$ is absolutely continuous on [0,1].

It follows from (D_1) and (D_2) that

ISSN (Online) 2249 - 054 X

International Journal of Computing and Corporate Research

Specialized and Refereed Journal for Research Scholars, Academicians, Engineers and Scientists

Volume 1 Issue 2 September 2011

where $x,y \in X$ ...(3.3)

Note that if (D_1) holds, then for every $x,y \in X$, the directional derivative is defined as follows:

$$f^{'}(x,\eta(y,x)) = D\phi(f,x,y;0) = \lim_{t \to 0} \frac{f(x+t\eta(y,x)) - f(x)}{t} \quad \dots(3.4)$$

<u>Theorem 3.2</u> Let X be an invex subset in a real vector space and $f : X \to R$, satisfy (D_1) and (D_2), the following assertions are then equivalent:

(a) f is pseudoinvex;

(b) for every integer kand every cycle $x_0, x_1, \dots, x_k, x_{k+1} = x_0$ in X, the inequality

$$\exists i \in \{0,\dots,k\} \ \exists f(x_i) :< f^{'}(x_i), \eta(x_{i+1}, x_i) > \ \geq 0$$

$$\Rightarrow < f^{'}(x_{i+1}), \eta(x_i, x_{i+1}) > \ \leq 0 \text{ hold;}$$

(c) for every $x,y \in X$, the inequality

$$\eta(y,x)^T f^{'}(x) \geq 0 \ \Rightarrow \ \eta(x,y)^T f^{'}(y) \leq 0 \text{ holds.}$$

4. Proofs

Proofs of theorem (3.1) observe that if f is $G\hat{a}teaux$ differentiable on x, then (D_1)holds and $f^{'}(x,\eta(y,x)) =< \eta(y,x), grad\ f(x) >$ for all $x,y \in X$, theorem(3.1) is then a direct consequence of theorem(3.2)

<u>Proof of theorem3.2</u>

ISSN (Online) 2249 - 054 X

International Journal of Computing and Corporate Research

Specialized and Refereed Journal for Research Scholars, Academicians, Engineers and Scientists

Volume 1 Issue 2 September 2011

(a)$\Rightarrow$(b). If (b) fails, then for some cycle $x_0, x_1, \ldots, x_{k+1}=x_0$ in x,

$$f^{'}(x_{i+1}, \eta(x_i, x_{i+1})) > 0, \; i = 0,1,\ldots,k$$

we have $D\phi(f, x_{i+1}, x_i; 0) > 0$, so

$$\phi(f, x_{i+1}, x_i; 0) < \phi(f, x_{i+1}, x_i; t) \quad \text{for small t} > 0 \; \ldots(4.1)$$

and as $\phi(f, x_{i+1}, x_i; .)$ is pseudoinvex on [0,1], it follows from (4.1) that

$$\phi(f, x_{i+1}, x_i; .) > \phi(f, x_{i+1}, x_i; .)$$

i.e. $f(x_{i+1}) > f(x_i)$**,** we obtain a contradictory chain of inequalities

$$f(x_0) < f(x_1) < \ldots < f(x_k) < f(x_{k+1}) = f(x_0)$$

and the contradiction means that (b) is true.

(b)$\Rightarrow$(c), obvious.

(c)$\Rightarrow$(a), suppose *f* is not pseudoinvex, There exist then $x, y \in X$ *, and* λ_0, $0 < \lambda_0 < 1$, such that

$$\text{f(x+}\lambda_0 \eta(y, x)) > f(y) \quad \ldots (4.2)$$

we claim that there exist λ_1 and λ_2, $0 < \lambda_1 < \lambda_0 < \lambda_2 < 1$, such that

$$D\phi(f, x, y; \lambda_1) > 0 \quad \ldots(4.3)$$

and $D\phi(f, y, x; 1-\lambda_2) > 0 \quad \ldots (4.4)$

But, if $D\phi(f, x, y; \lambda) \leq 0$ for all $\lambda, 0 < \lambda < \lambda_0$, then with (3.3) and the identity

ISSN (Online) 2249 - 054 X

International Journal of Computing and Corporate Research

Specialized and Refereed Journal for Research Scholars, Academicians, Engineers and Scientists

Volume 1 Issue 2 September 2011

$\phi(f,x,(y+\lambda_0\eta(y,x):\lambda)=\phi(f,x,y;\lambda_0\lambda)$

we obtain,

$$f(x+\lambda_0\eta(y,x))=f(x)+\int_0^1 D\phi(f,x,x+\lambda_0\eta(y,x);\lambda)d\lambda$$

$$=f(x)+\int_0^1 D\phi(f,x,y;\lambda_0\lambda)d\lambda$$

$$=f(x)+\frac{1}{\lambda_0}\int_0^{\lambda_0} D\phi(f,x,y;r)dr\le f(x)$$

which contradicts (4.2),

then if $D\phi(f,y,x;\lambda)\le 0 \quad \forall\lambda \ \& \ 0<\lambda<1-\lambda_2$, then

$$f(x+\lambda_0\eta(y,x))=f(y)+\int_0^1 D\phi(f,y,x+\lambda_0\eta(y,x);\lambda)d\lambda$$

$$=f(y)+\frac{1}{1-\lambda_0}\int_0^{1-\lambda_0} D\phi(f,y,x,r))dr\le f(y)$$

which again contradicts (4.2). The claim is thus proved.

Set how $\quad x_k=(1-\lambda_k)x+\lambda_k y; \qquad k=1,2$

And using (4.3) and (4.4) and the identities,

$$\phi(f,x_1,x_2;\lambda)=\phi(f,x,y;\lambda_1+\lambda(\lambda_2-\lambda_1))$$

$$\phi(f,x_1,x_2;\lambda)=\phi(f,y,x;1-\lambda_2+\lambda(\lambda_2-\lambda_1))$$

ISSN (Online) 2249 - 054 X

International Journal of Computing and Corporate Research

Specialized and Refereed Journal for Research Scholars, Academicians, Engineers and Scientists

Volume 1 Issue 2 September 2011

hence

$$f^{'}(x_1,\eta(x_2,x_1)) = D\phi(f,x_1,x_2,0) = (\lambda_2-\lambda_1)D\phi(f,x,y;\lambda_1) > 0$$

$$f^{'}(x_2,\eta(x_1,x_2)) = D\phi(f,x_2,x_1,0) = (\lambda_2-\lambda_1)D\phi(f,y,x;1-\lambda_2) > 0$$

Hence a contradiction with (c).

References

1. A.Daniilidis and N.Hadjisavvas, On the subdifferentials of quasiconvex and pseudoconvex functions and cyclic monotonicity, Journal of Mathematical Analysis and Application ,237(1999),30-42.
2. D.Aussel, J.N.Corvellec and M.Lassonde , Subdifferential charecterization of quasiconvexity and convexity, Journal of Convex Analysis , Vol.1,No.2(1994),195-201.
3. R.I.Kachurovaskii , On monotone operators and convex functionals, Upsekhi Matem Nauk , Vol.15;4,(1960),213-215: Uchenye Zapiski, Moskovskogo obtastn , Pedagog , Inst. 1962,231-243,Russian.
4. R.I.Kachurovaskii , Non –linear operators of bounded variation, monotone and convex operators in Banach space, Upsekhi Matem Nauk , Vol.21;.5,(1966),256-257,Russian.
5. R.I.Kachurovaskii , Non –linear operators in Banach space, Upsekhi Matem Nauk , Vol.23;.2,(1968),121-168,Russian.
6. R.T.Rockafellar , characterization of the subdifferentials of the convex functions, Pacific J.Math , Vol.17(1966),497-510.

ISSN (Online) 2249 - 054 X

International Journal of Computing and Corporate Research

Specialized and Refereed Journal for Research Scholars, Academicians, Engineers and Scientists

Volume 1 Issue 2 September 2011

7. R.T.Rockafellar , convex functions monotone operators and variational inequalities in: Theory and Applications of monotone operators, A.Ghizzetti,1969,35-65,Tipografia oderisi Editrice ,Gubbio,Italy.

8. R.T.Rockafellar , Convex Analysis,Princeton University Press,1970.

9. R.T.Rockafellar , On the maximal monotonicity of subdifferential mappings, Pacific J.Math , Vol.33(1970),209-216.

10. S.karlin , Mathematical methods and theory in games, programming and Economics, Pergamon Press, London-Paris,1959.

11. V.L.Levin , Reduced cost functions and their applications, Journal of Mathematical Economics (Accepted).

12. V.L.Levin , Quasi-convex functions and quasi-monotone operators , Journal of Convex Analysis Vol.2, No.1/2 (1995),167-172.

13. X.M.Yang, X.Q.Yang and K.L.Teo , Generalized invexity and generalized inmonicity, J. Optim. Theory Appl. (2001), Accepted.