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PSEUDOINVEX FUNCTIONS AND PSEUDOINMON OPERATORS

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Abstract

The notions of a pseudoinvex functions and pseudoinmon operator are introduced, with cyclic pseudoinmon operator also. In this paper, we are studying the relation among pseudoinvex functions. Pseudoinmon and cyclic pseudoinmon operators.

Key words: Pseudoinvex functions, Pseudoinmon operator, Cyclic Pseudoinmon operator, Pseudomonotone, Cyclic Pseudomonotone.

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1.Introduction:

The notion of a (not necessary single-valued) monotone operator was first proposed by Kachurovaskii[3], interested reader may see[4,5] , cyclically monotone operator was introduced and investigated by Rockafellar[6-9].

For a given smooth function f on a convex domain characterization of its convexity in terms of monotonicity or cyclically monotonicity of $\text{grad } f$ are known as follows:

$$f \text{ is convex} \Leftrightarrow \text{grad } f \text{ is monotone} \Leftrightarrow \text{grad } f \text{ is cyclically monotone} \quad \text{---- (A)}$$

Levin [12], get similar characterization theorems connecting

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f is quasiconvex $\Leftrightarrow \text{grad } f$ is quasimonotone $\Leftrightarrow \text{grad } f$ is quasicyclically monotone- (B)

The term inmonicity was introduced in [13] and showed that

Pseudoinmonicity \Rightarrow quasiinmonicity, but converse is not true. ----- (C)

On the other hand invexity of a real-valued function, the relation in

Pseudoinvexity \Rightarrow quasiinvexity, but converse is not true. ----- (D)

In [12], Levin showed that such operators are closely related to the so-called demand functions in mathematical economics. It is worth noting that monotonicity has played a very important role in the existence and solution methods of variational inequality problems. A plenty of applications of solutions of variational inequality problems are present in the convexity literature.

Our aim of present paper is to get characterization theorems connecting pseudoinvex functions with pseudoinmon and cyclically pseudoinmon operators.

2. Notations and Preliminaries:

Let W be a Hausdorff locally convex space, X a convex set in it, and w^* the dual space.

Definition 2.1 [12] An operator $f : X \rightarrow w^*$ is Pseudomonotone if

$$\langle y - x, f(x) \rangle \geq 0 \Rightarrow \langle y - x, f(y) \rangle \geq 0 \quad \dots \dots \dots (1.1)$$

$$\forall x, y \in X$$

Definition 2.2 [12] An operator $f : x \rightarrow w^*$ is cyclically pseudomonotone if

$$\langle x_{i+1} - x_i, f(x_i) \rangle > 0 \quad \forall i = 1, 2, \dots, k$$



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$$\Rightarrow \langle x_{j+1} - x_j, f(x_j) \rangle < 0 \quad \text{for some } j = 1, 2, \dots, k$$

Definition 2.3 [13] A differentiable operator f on a subset $X \subseteq R^n$ is a Pseudoinvex with respect to $\eta : X \times X \rightarrow X$ if for every pair of distinct points $x, y \in X$

$$\eta(y, x)^T f'(x) \geq 0 \Rightarrow f(y) \geq f(x).$$

Definition 2.4 [13] Let $X \subseteq R^n$ be an invex set with respect to $\eta : X \times X \rightarrow X$ and $f : X \rightarrow R$ is pseudoinvex on X if $\forall x, y \in X$.

$$\eta(y, x)^T f'(x) \geq 0 \Rightarrow \eta(y, x)^T f'(y) \leq 0.$$

The goal of the present paper is to get characterization theorem connecting Pseudo invex functions, Pseudo invex and cyclically Pseudo invex operators on an invex domain.

3. Characterization Theorem:

Let X be an invex set in a real vector space.

Definition 3.1 Let $X \subseteq R^n$ be an invex set with respect to $\eta : X \times X \rightarrow X$, $f : X \rightarrow R$ is said to cyclically pseudoinvex with respect to $\exists i = \{1, 2, \dots, k\}$, $\exists f(x_i) : \langle f(x_i), \eta(x_{i+1}, x_i) \rangle \geq 0$

$$\langle f(x_{i+1}), \eta(x_i, x_{i+1}) \rangle \leq 0$$

Let us define.

$$\phi(f, x, y; t) = f(x + t\eta(y, x)) \quad \dots (3.1)$$



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Theorem 3.1 Let X be an invex open set in a Hausdorff locally convex space w , and suppose that a function $f : X \rightarrow R$ has two properties as follows:

- (i) f is $G\hat{a}teaux$ differentiable on X , i.e. every $x \in X$, there exist an element $grad f(x) \in w^*$ such that

$$\lim_{\Delta t \rightarrow 0} \frac{f(x + t\eta(y,x)) - f(x)}{t} = \langle \eta(y,x), grad f(x) \rangle \text{ for all } \eta(y,x) \in E.$$

- (ii) for every $x, y \in X$, the function $\phi(f, x, y; \cdot)$ given by (3.1) is absolutely continuous on $[0, 1]$.

The following assertions are then equivalent:

- (a) f is pseudoinvex,
- (b) the operator $grad f$ is cyclically pseudoinmon
- (c) The operator $grad f$ is pseudoinmon.

Before to pass on to a more general characterization theorem, let us formulate two assumption (D_1) and (D_2) , on a function $f : X \rightarrow R$, where X is an invex set in a real vector space. The assumptions are expressed in terms of the functions $\phi(f, x, y; \cdot)$ (see (3.1)) as follows:

(D_1) for every $x, y \in X$ and every $t, 0 \leq t \leq 1$, there exist the right derivative

$$D \phi(f, x, y; t) = \lim_{\Delta t \rightarrow 0} \frac{\phi(f, x, y; t + \Delta t) - \phi(f, x, y; t)}{\Delta t} \text{ ----- (3.2)}$$

(D_2) for every $x, y \in X$, $\phi(f, x, y; \cdot)$ is absolutely continuous on $[0, 1]$.

It follows from (D_1) and (D_2) that



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where $x, y \in X$... (3.3)

Note that if (D_1) holds, then for every $x, y \in X$, the directional derivative is defined as follows:

$$f'(x, \eta(y, x)) = D\phi(f, x, y; 0) = \lim_{t \rightarrow 0} \frac{f(x + t\eta(y, x)) - f(x)}{t} \dots (3.4)$$

Theorem 3.2 Let X be an invex subset in a real vector space and $f : X \rightarrow R$, satisfy (D_1) and (D_2) , the following assertions are then equivalent:

- (a) f is pseudoinvex;
- (b) for every integer k and every cycle $x_0, x_1, \dots, x_k, x_{k+1} = x_0$ in X , the inequality

$$\exists i \in \{0, \dots, k\} \exists f(x_i) : \langle f'(x_i), \eta(x_{i+1}, x_i) \rangle \geq 0$$

$$\Rightarrow \langle f'(x_{i+1}), \eta(x_i, x_{i+1}) \rangle \leq 0 \text{ hold;}$$

- (c) for every $x, y \in X$, the inequality

$$\eta(y, x)^T f'(x) \geq 0 \Rightarrow \eta(x, y)^T f'(y) \leq 0 \text{ holds.}$$

4. Proofs

Proofs of theorem (3.1) observe that if f is $G \hat{a} teaux$ differentiable on x , then (D_1) holds and $f'(x, \eta(y, x)) = \langle \eta(y, x), grad f(x) \rangle$ for all $x, y \in X$, theorem(3.1) is then a direct consequence of theorem(3.2)

Proof of theorem 3.2



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(a) \Rightarrow (b). If (b) fails, then for some cycle $x_0, x_1, \dots, x_{k+1}=x_0$ in X ,

$$f'(x_{i+1}, \eta(x_i, x_{i+1})) > 0, \quad i = 0, 1, \dots, k$$

we have $D\phi(f, x_{i+1}, x_i; 0) > 0$, so

$$\phi(f, x_{i+1}, x_i; 0) < \phi(f, x_{i+1}, x_i; t) \quad \text{for small } t > 0 \dots (4.1)$$

and as $\phi(f, x_{i+1}, x_i; \cdot)$ is pseudoinvex on $[0, 1]$, it follows from (4.1) that

$$\phi(f, x_{i+1}, x_i; \cdot) > \phi(f, x_{i+1}, x_i; \cdot)$$

i.e. $f(x_{i+1}) > f(x_i)$, we obtain a contradictory chain of inequalities

$$f(x_0) < f(x_1) < \dots < f(x_k) < f(x_{k+1}) = f(x_0)$$

and the contradiction means that (b) is true.

(b) \Rightarrow (c), obvious.

(c) \Rightarrow (a), suppose f is not pseudoinvex, There exist then $x, y \in X$, and $\lambda_0, 0 < \lambda_0 < 1$, such that

$$f(x + \lambda_0 \eta(y, x)) > f(y) \quad \dots (4.2)$$

we claim that there exist λ_1 and λ_2 , $0 < \lambda_1 < \lambda_0 < \lambda_2 < 1$, such that

$$D\phi(f, x, y; \lambda_1) > 0 \quad \dots (4.3)$$

and $D\phi(f, y, x; 1 - \lambda_2) > 0 \quad \dots (4.4)$

But, if $D\phi(f, x, y; \lambda) \leq 0$ for all $\lambda, 0 < \lambda < \lambda_0$, then with (3.3) and the identity



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$$\phi(f, x, (y + \lambda_0 \eta(y, x)) : \lambda) = \phi(f, x, y; \lambda_0 \lambda)$$

we obtain,

$$\begin{aligned} f(x + \lambda_0 \eta(y, x)) &= f(x) + \int_0^1 D\phi(f, x, x + \lambda_0 \eta(y, x); \lambda) d\lambda \\ &= f(x) + \int_0^1 D\phi(f, x, y; \lambda_0 \lambda) d\lambda \\ &= f(x) + \frac{1}{\lambda_0} \int_0^{\lambda_0} D\phi(f, x, y; r) dr \leq f(x) \end{aligned}$$

which contradicts (4.2),

then if $D\phi(f, y, x; \lambda) \leq 0 \quad \forall \lambda \text{ \& } 0 < \lambda < 1 - \lambda_2$, then

$$\begin{aligned} f(x + \lambda_0 \eta(y, x)) &= f(y) + \int_0^1 D\phi(f, y, x + \lambda_0 \eta(y, x); \lambda) d\lambda \\ &= f(y) + \frac{1}{1 - \lambda_0} \int_0^{1 - \lambda_0} D\phi(f, y, x, r) dr \leq f(y) \end{aligned}$$

which again contradicts (4.2). The claim is thus proved.

Set how $x_k = (1 - \lambda_k)x + \lambda_k y; \quad k = 1, 2$

And using (4.3) and (4.4) and the identities,

$$\phi(f, x_1, x_2; \lambda) = \phi(f, x, y; \lambda_1 + \lambda(\lambda_2 - \lambda_1))$$

$$\phi(f, x_1, x_2; \lambda) = \phi(f, y, x; 1 - \lambda_2 + \lambda(\lambda_2 - \lambda_1))$$



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hence

$$f'(x_1, \eta(x_2, x_1)) = D\phi(f, x_1, x_2, 0) = (\lambda_2 - \lambda_1)D\phi(f, x, y; \lambda_1) > 0$$

$$f'(x_2, \eta(x_1, x_2)) = D\phi(f, x_2, x_1, 0) = (\lambda_2 - \lambda_1)D\phi(f, y, x; 1 - \lambda_2) > 0$$

Hence a contradiction with (c).

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